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1983 J. Phys. A: Math. Gen. 16 3885

(http://iopscience.iop.org/0305-4470/16/16/026)

J. Phys. A: Math. Gen. **16** (1983) 3885–3984. Printed in Great Britain

Bounds on self-avoiding walks on directed square lattices

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Received 4 January 1983, in final form 21 June 1983

Abstract. The Manhattan lattice is the covering lattice of the L lattice, and we have studied self-avoiding walks on this pair of lattices. Upper and lower bounds on the connective constant (μ) have been obtained, as have series analysis estimates. We find that

$$1.5353 < \mu \approx 1.5658 \pm 0.0010 < 1.5986 \qquad \text{(L lattice)}$$

and

$$1.6336 < \mu \approx 1.7340 \pm 0.0015 < 1.7912 \qquad \text{(Manhattan)}.$$

1. Introduction

We have studied self-avoiding random walks (SAWS) on a pair of related lattices, both being restrictions of the square lattice. The lattices are the Manhattan lattice and the L lattice, and these are illustrated in figure 1. The Manhattan lattice has alternate rows (or columns) parallel, and adjacent rows (or columns) antiparallel, while the L lattice has the property that each bond on a path must be at right angles to its predecessor.

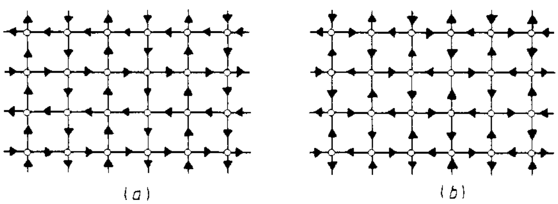

Figure 1. (*a*) Manhattan lattice, (*b*) L lattice.

As a consequence, SAWS on the L lattice may be viewed as a concatenation of L shaped, two-step, walks. This latter lattice has been variously named in the literature as the two-choice 90° lattice (Wall *et al* 1955) and the underlying lattice of the Manhattan square lattice (Kasteleyn 1963), but we prefer the simpler notation of the L lattice.

Self-avoiding walks on these lattices are of particular interest for several reasons. Firstly, as pointed out by Kasteleyn (1963), the Manhattan graph (lattice) is the covering of the L graph, and the L graph is the only closed, oriented, square lattice whose covering is a closed, oriented square lattice. This observation led Kasteleyn to prove that the number of Hamiltonian cycles, N_H, on the P-site closed Manhattan lattice (with periodic boundary conditions) was given by

$$N_{\mathrm{H}}(P) = C\lambda^{P}[1 + \mathrm{O}(P^{-\alpha})] \tag{1.1}$$

where $\lambda = \exp(g/\pi) = 1.3381\ldots$, and $g =$ Catalan's constant. The asymptotic form (1.1) was in fact found by Barber, who showed that $C = 2(\frac{1}{2}\theta_2\theta_3\theta_4)^{4/3} = 0.696\,60\ldots$ where $\theta_i = \theta_i(0, \mathrm{e}^{-\pi})$ is an elliptic theta function, and $0 < \alpha < 1$. Note that this asymptotic behaviour is different to, and indeed simpler than, that expected to prevail for SAWs. In that case, it is believed that $c_n \sim A\mu^n n^g$, where c_n is the cardinality of n-step SAWs and $g = \gamma - 1 > 0$. For the L lattice Malakis (1975) numerically estimated the number of Hamiltonian cycles for that lattice (with different boundary conditions) and found $N_{\mathrm{H}}(P) \sim \lambda_{\mathrm{L}}^{P}$ with $\lambda_{\mathrm{L}} \approx 1$.

Secondly, Nienhuis (1982) has recently obtained the exact connective constant for the SAW problem on the honeycomb lattice, and has given convincing arguments to suggest that the growth exponent g, defined above, is exactly $\frac{11}{32}$. Nienhuis's technique is restricted in its application to those lattices for which the graphs contributing to the zero field free-energy are disconnected polygons. This in effect restricts the coordination number to three, so that Nienhuis's technique might be applicable to the Manhattan and L lattices. Note that there is no known proof that the exponent g for the L and Manhattan lattices should be the same as that for the unrestricted square lattice, but universality suggests strongly that this should be so. Malakis (1975) has also discussed this point, and gives several arguments, as well as numerical evidence, in support of this belief. We subsequently study the relevant series but for the moment assert that this equality can be assumed with a fair degree of confidence.

The third reason that these lattices are particularly interesting relates to the proof by Hammersley and Welsh (1962) that

$$c_n \sim A\mu^n \exp(\mathrm{O}(\sqrt{n})).$$

The growth term, $\exp(\mathrm{O}(\sqrt{n}))$, is derived by an unfolding transformation and follows from the number of unequal partitions of the integers. For these restricted square lattices, and particularly the L lattice, the number of such transformations is far fewer, and it is to be hoped that a careful study of unfolding transformations on these lattices might enable the term $\exp(\mathrm{O}(\sqrt{n}))$ to be sharpened, so that it may approach more closely the widely believed result $\exp(\mathrm{O}(\log n))$. At the time of writing, however, we have only been able to sharpen the constant multiplying the $n^{1/2}$ term.

In the remainder of this paper we establish quite tight rigorous bounds on the connective constant for the L lattice, weaker bounds for the Manhattan lattice, and numerical estimates for these quantities.

The lower bounds are found by extending a theorem of Kesten (1963), so that it applies to non-regular lattices, and then enumerating a subset of SAWs (bridges) as defined in § 2. Upper bounds are obtained by the method of Fisher and Sykes (1959) and by a simple extension of a method due to Ahlberg and Janson (1982).

2. Lower bounds

Kesten's theorem for lower bounds applies to the general d-dimensional ($d \geq 2$) hypercubic lattice, but for our purposes we will discuss only the two-dimensional case. Consider the square lattice to be the integer points of a Cartesian coordinate system. We denote by Γ_n the set of n-step SAWs with origin given by the origin of the coordinate system, while c_n denotes the cardinality of the set. Then the set B_n, denoting n-step *bridges* with cardinality b_n, is defined as a subset of Γ_n such that members of B_n satisfy

$$0 = X_0(c_n) < X_i(c_n) \leqslant X_n(c_n), \qquad 1 \leqslant i \leqslant n,$$

where $X_i(c_n)$ is the X coordinate of the ith monomer in the particular n-step SAW c_n. Thus $B_n \subseteq \Gamma_n$ and $b_n \leqslant c_n$. These walks can be visualised as the set of SAWs whose X coordinates are all strictly positive (beyond the origin point) and whose end point has maximal (but not necessarily unique) X coordinate. The set Λ_n of *irreducible bridges*, with cardinality λ_n, can be defined as a subset of the bridges satisfying the additional constraint that no $k < n$ exists for which $0 < X_i(c_n) \leqslant X_k(c_n) < X_j(c_n) \leqslant X_n(c_n)$ for all i and j such that $0 < i \leqslant k < j \leqslant n$. Irreducible bridges are therefore those bridges which cannot be viewed as the concatenation of two bridges.

In terms of these definitions, Kesten (1963) proved the lemma

$$b_n = \sum_{k=1}^{n} \lambda_n b_{n-k}, \qquad n = 1, 2, \ldots, \tag{2.1}$$

and consequently for $|x| < 1/\mu$,

$$B(x) = \sum_{n=0}^{\infty} b_n x^n = \frac{1}{1 - \Lambda(x)} \tag{2.2}$$

where

$$b_0 = 1 \qquad \text{and} \qquad \Lambda(x) = \sum_{k=1}^{\infty} \lambda_k x^k. \tag{2.3}$$

Further, Kesten proved that the connective constant μ can be obtained from the result that μ^{-1} is the unique positive root of $\Lambda(x) = 1$, and hence lower bounds on μ can be obtained from the solution of the polynomial equation

$$\sum_{n=1}^{N} \lambda_n \mu_N^{-n} = 1, \tag{2.4}$$

whose unique, positive root satisfies $\mu_N \leqslant \mu$. These results of Kesten cannot be directly applied to the L or the Manhattan lattice, due to their non-regular nature, but the following result can be obtained. Firstly observe that the first step of any bridge is horizontal. Now divide bridges into two classes. Members of class A have an odd number of steps while members of class B have an even number of steps. We use the notation of the previous paragraphs with superscripts A and B denoting the two classes. The analogue of Kesten's lemma *for the* L *lattice only* is then

$$b_n^{\mathrm{B}} = \sum_{k=1}^{n/2} \lambda_{2k}^{\mathrm{B}} b_{n-2k}^{\mathrm{B}} \qquad n = 2, 4, 6, \ldots \tag{2.5a}$$

$$b_n^{\mathrm{A}} = \sum_{k=0}^{[n/2]} \lambda_{2k+1}^{\mathrm{A}} b_{n-2k-1}^{\mathrm{B}} \qquad n = 1, 3, 5, \ldots. \tag{2.5b}$$

The proof of this lemma follows, *mutatis mutandis*, from Kesten's proof for the general hypercubic lattice. From those results follow

$$B^{\mathrm{B}}(x) = \sum_{n=0}^{\infty} b_{2n}^{\mathrm{B}} x^{2n} = \frac{1}{1 - \Lambda^{\mathrm{B}}(x)} \tag{2.6a}$$

$$B^{A}(x)=\sum_{n=0}^{\infty} b_{2n+1}^{A} x^{2n+1}=\frac{\Lambda^{A}(x)}{1-\Lambda^{B}(x)} \tag{2.6b}$$

where $b_0=1$, $\Lambda^{A}(x)=\Sigma_{n=0}^{\infty}\lambda_{2n+1}^{A}x^{2n+1}$ and $\Lambda^{B}(x)=\Sigma_{n=1}^{\infty}\lambda_{2n}^{B}x^{2n}$. Adding (2.6$a$) and (2.6$b$) gives the fundamental equation

$$B(x)=(1-\Lambda^{A}(x))/(1-\Lambda^{B}(x)). \tag{2.7}$$

The connective constant, μ, can be obtained from the result that μ^{-1} is the unique, positive root of $\Lambda^{B}(x)=1$, and hence lower bounds on μ can be obtained from the solution of the polynomial equation

$$\sum_{n=1}^{N}\lambda_{2n}^{B}\mu_{N}^{-2n}=1 \tag{2.8}$$

whose unique, positive root satisfies $\mu_N \leqslant \mu$.

In order to utilise these results, we have generated all bridges up to and including those of 30 steps. This was done by generating all walks up to 30 steps, and identifying bridges from the walks.

In this way the bridge generating function was found to be

$$\begin{aligned} B(x)/x = {} & 1+2x+2x^2+4x^3+4x^4+8x^5+10x^6+18x^7+22x^8+40x^9+50x^{10}+90x^{11} \\ & +116x^{12}+210x^{13}+268x^{14}+486x^{15}+628x^{16}+1140x^{17}+1474x^{18} \\ & +2684x^{19}+3474x^{20}+6328x^{21}+8210x^{22}+14\,976x^{23}+19\,484x^{24} \\ & +35\,564x^{25}+46\,278x^{26}+84\,532x^{27}+110\,238x^{28}+201\,448x^{29} \\ & +263\,050x^{30}+\ldots . \end{aligned} \tag{2.9}$$

Using (2.6a) we obtained the irreducible bridge generating function for class B bridges, $\Lambda^{B}(x)$, which was

$$\begin{aligned} \Lambda^{B}(x) = {} & 2x^2+2x^8+2x^{12}+10x^{14}+2x^{16}+28x^{18}+44x^{20}+64x^{22}+204x^{24} \\ & +412x^{26}+720x^{28}+2016x^{30}+\ldots . \end{aligned} \tag{2.10}$$

Numerical solution of the polynomial $\Lambda^{B}(x)=1$ gives the bound $\mu>1.5353$. For the Manhattan lattice we find that the fundamental equation (2.7) holds with superscripts A and B reversed. That is,

$$B(x)=(1+\Lambda^{B}(x))/(1-\Lambda^{A}(x)). \tag{2.11}$$

Hence a lower bound is identified with the unique positive root of the polynomial

$$\sum_{n=1}^{N}\lambda_{2n-1}^{A}\mu_{N}^{-2n+1}=1. \tag{2.12}$$

Primitive enumeration by hand gives the first few coefficients of Λ^{A} as $\Lambda^{A}(x)=x+x^3+x^5+x^7+2x^9+\ldots$. Equation (2.12) then gives the bound $\mu>1.6217$. Automating the counting procedure would enable this bound to be sharpened considerably, but having demonstrated the method for the L lattice, we have not proceeded with the enumerations for the Manhattan lattice. Nevertheless, a slight improvement is possible by observing that, since $\lambda_{2n-1}^{A}\geqslant 0$ for all $n\geqslant 1$, the polynomial (2.12) with λ_{2n-1}^{A} replaced by λ_{2n-1}^{*} where λ_{2n-1}^{*} satisfies $0\leqslant\lambda_{2n-1}^{*}\leqslant\lambda_{2n-1}^{A}$ also provides a lower bound. Then the observation that $\lambda_{2n-1}^{A}\geqslant 1$ (equality is achieved by considering the irreducible bridge whose first step is to the right and all of whose subsequent steps are in the y direction) means that we can write $\Lambda^{A}(x) \geq x + x^3 + x^5 + x^7 + 2x^9 + x^{11} + x^{13} + x^{15} + \ldots = x/(1-x^2) + x^9$, for $x \geq 0$ and equating the right-hand side to unity gives the bound $\mu > 1.6272$. Finally the observation that $\lambda_{11}^{A} \geq 3$ improves this bound to $\mu > 1.6336$.

3. Upper bounds

We have used two methods to obtain upper bounds. The first method, due to Wakefield (1951) and subsequently extensively used by Fisher and Sykes (1959), consists of generating the set of random walks with the restriction that closures of less than k steps are forbidden. For finite k the problem is now Markovian in nature, and in the limit of large k it reduces to the SAW problem. Denoting the number of such restricted random walks as $c_n^{(k)}$, it is clear that $c_n^{(k)} \geq c_n$, where c_n is, as before, the number of n-step SAWs and

$$\lim_{n\to\infty} \frac{1}{n} \ln c_n^{(k)} \geq \lim_{n\to\infty} \frac{1}{n} \ln c_n = \ln \mu. \tag{3.1}$$

As the problem is Markovian, the left-hand limit can be evaluated exactly, and upper bounds obtained. Following Fisher and Sykes, and making modifications appropriate to the peculiarities of the L lattice, the elimination of four-step closures (squares) gives rise to a 3×3 matrix the unique positive eigenvalue of which is $\frac{1}{2}(1+\sqrt{5}) = 1.618\ldots$, so that $\mu \leq 1.618\ldots$. This is quite a good bound, as the topology of the lattice ensures that the next smallest closure can only occur at 12 steps. The resulting 15×15 matrix has as its largest eigenvalue 1.611, and so $\mu \leq 1.611$, an improvement over the earlier result of less than $\frac{1}{2}\%$.

An alternative method makes use of an inequality given by Ahlberg and Janson (1982). Following them, we make the following definition. If $\gamma \in \Gamma_k$, where $k \leq n$, let c_n^{γ} denote the number of SAWs (of length n) that begin with γ. That is, γ is the first k steps of the n-step SAW c_n^{γ}. Then

$$c_n = \sum_{\gamma\in\Gamma_k} c_n^{\gamma}, \qquad k \leq n. \tag{3.2}$$

Now let $m, n \geq k$. Any path of length $m+n-k$ can be considered as two SAWs, one of length m and one of length n, overlapping for k steps. Sorting all walks in Γ_{m+n-k} according to the overlapping part, Ahlberg and Janson obtain the important inequality

$$c_{m+n-k} \leq \sum_{\Gamma_k} c_n^{\gamma} c_m^{\gamma^*} \tag{3.3}$$

where γ^* is γ reversed. Now for the L lattice, and the honeycomb lattice (but not the Manhattan lattice or most other lattices), all two-step SAWs are identical. That is, Γ_2 consists of $c_2 = 4$ identical paths (for the L lattice), all of which are L shaped paths. Thus if $\gamma = \Gamma_2$ we have $c_n^{\gamma} = c_n/c_2$ and the above inequality becomes

$$c_{m+n-2} \leq \sum_{\Gamma_2} c_n^{\gamma} c_m^{\gamma^*} = c_n c_m / c_2.$$

To proceed further, we use a theorem on sub-additive functions due to Wilker and Whittington (1979).

Theorem. If $b_n \cdot b_m \geqslant b_{n+f(m)}$ for a positive function f satisfying $\lim_{m\to\infty} f(m)/m = 1$, then $\mu = \lim_{n\to\infty} n^{-1} \ln b_n$ and $b_n \geqslant \mu^{f(n)}$.

Application of this theorem then gives the result (with $f(m) = m-2$) that $\mu \leqslant (c_n/c_2)^{1/(n-2)}$. Very recently Grassberger (1982) has extended the SAW series for the L lattice. His highest coefficient is $c_{44}/4 = 360\,311\,379$ giving the bound $\mu \leqslant 1.5986$. For the honeycomb lattice we have $c_{34} = 4\,531\,816\,950$, which gives the bound $\mu < 1.8943$ in agreement with Ahlberg and Janson. (The exact result is $\mu = 1.847\,759\ldots$ (Nienhuis 1982).) For the Manhattan lattice, the two-step walks are not symmetric, and we use the weaker inequality $\mu \leqslant (c_n/c_1)^{1/(n-1)}$ plus the coefficient $c_{28} = 13\,687\,192$ (Malakis 1975) to obtain the bound $\mu < 1.7912$.

4. Series analysis

We have reanalysed the longest extant series for the Manhattan lattice (Malakis 1975) and analysed the series given by Grassberger (1982) for the L lattice. It is known from studies of the Ising model that series on low coordination number lattices, such as the honeycomb, display a four-term periodicity in their ratios, due to the presence of four singularities at $\pm v_c$ and $\pm i v_c$, where $v_c = \tanh(J/kT_c)$. For the SAW problem on the honeycomb lattice the series behave similarly, and Guttmann and Whittington (1978) have shown that a singularity at $-1/\mu$ can be expected as well as the physical singularity at $1/\mu$. While we have not been able to establish the presence of singularities at $\pm i/\mu$, the series behaviour leads us to believe that a conjugate pair of singularities does exist at a position close to $\pm i/\mu$. Preliminary analysis of the Manhattan and L lattice series also shows evidence of a four-term periodicity among the coefficients (particularly evident in Dlog Padé approximants) and we have analysed the series accordingly. If we form the ratios of every fourth term, and take the fourth root of that ratio, we obtain four distinct sequences each of which should converge to μ. That is, we define

$$r_n^{(\alpha)} = [c_{4n+\alpha}/c_{4n+\alpha-4}]^{1/4}, \qquad \alpha = 0, 1, 2, 3.$$

Each sequence $\{r_n^{(\alpha)}\}$ is then extrapolated linearly and quadratically by forming linear extrapolants $l_n^{(\alpha)} = n r_n^{(\alpha)} - (n-1) r_{n-1}^{(\alpha)}$ and quadratic extrapolants $q_n^{(\alpha)} = \frac{1}{2}[n l_n^{(\alpha)} - (n-2) l_{n-1}^{(\alpha)}]$.

We illustrate the method in table 1 below for the honeycomb lattice. Focusing on the latter entries in the table, we see that the ratios $r_n^{(\alpha)}$ are uniformly decreasing, the linear extrapolants with the exception of $l_n^{(0)}$ are also uniformly decreasing, while the quadratic extrapolants $q_n^{(0)}$ and $q_n^{(1)}$ are increasing towards the known limit of $1.847\,759\ldots$, while $q_n^{(2)}$ and $q_n^{(3)}$ are decreasing towards this limit. Averaging the last two entries of $q_n^{(\alpha)}$, $\alpha = 0, 1, 2, 3$, we obtain 1.847 71 and quote confidence limits of ± 0.0009, which estimate contains all the last eight quadratic extrapolants used in the averaging.

For the Manhattan lattice, we obtained the results shown in table 2. Again the ratios $r_n^{(\alpha)}$ are monotonically decreasing, but the linear extrapolants $l_n^{(0)}$ and $l_n^{(1)}$ are increasing, while $l_n^{(2)}$ and $l_n^{(3)}$ are decreasing. These trends alone, if bounding, yield $1.7331 < \mu < 1.7347$. The central value of 1.7340 is well supported by the quadratic extrapolants, and we estimate

$$\mu_{\text{Manhattan}} = 1.7340 \pm 0.0015.$$

Table 1. Ratio analysis of honeycomb lattice SAW generating function.

n	$r_n^{(0)}$	$l_n^{(0)}$	$q_n^{(0)}$	$r_n^{(1)}$	$l_n^{(1)}$	$q_n^{(1)}$	$r_n^{(2)}$	$l_n^{(2)}$	$q_n^{(2)}$	$r_n^{(3)}$	$l_n^{(3)}$	$q_n^{(3)}$
2	1.9343	1.6553	—	1.9168	1.8337	—	1.9180	1.8680	—	1.9125	1.8737	—
3	1.9040	1.8434	1.9374	1.8968	1.8567	1.8683	1.8972	1.8556	1.8494	1.8930	1.8538	1.8439
4	1.8893	1.8453	1.8471	1.8863	1.8549	1.8531	1.8853	1.8495	1.8434	1.8826	1.8516	1.8494
5	1.8809	1.8474	1.8505	1.8791	1.8500	1.8426	1.8780	1.8489	1.8478	1.8763	1.8512	1.8506
6	1.8753	1.8470	1.8463	1.8740	1.8485	1.8456	1.8731	1.8488	1.8488	1.8720	1.8503	1.8485
7	1.8712	1.8470	1.8469	1.8703	1.8482	1.8473	1.8696	1.8486	1.8482	1.8688	1.8496	1.8479
8	1.8682	1.8471	1.8475	1.8675	1.8480	1.8476	1.8670	1.8485	1.8479			

Table 2. Ratio analysis of Manhattan lattice SAW generating function.

n	$r_n^{(0)}$	$l_n^{(0)}$	$q_n^{(0)}$	$r_n^{(1)}$	$l_n^{(1)}$	$q_n^{(1)}$	$r_n^{(2)}$	$l_n^{(2)}$	$q_n^{(2)}$	$r_n^{(3)}$	$l_n^{(3)}$	$q_n^{(3)}$
2	1.8212	1.7080	—	1.8083	1.7177	—	1.7695	1.7318	—	1.7883	1.7554	—
3	1.7886	1.7234	1.7311	1.7824	1.7307	1.7372	1.7779	1.7407	1.7452	1.7730	1.7425	1.7360
4	1.7727	1.7252	1.7269	1.7700	1.7325	1.7344	1.7672	1.7350	1.7293	1.7645	1.7389	1.7353
5	1.7643	1.7304	1.7384	1.7625	1.7327	1.7328	1.7608	1.7350	1.7351	1.7591	1.7374	1.7352
6	1.7588	1.7314	1.7332	1.7576	1.7331	1.7340	1.7564	1.7347	1.7341	1.7553	1.7362	1.7338
7	1.7550	1.7322	1.7342									

Table 3. Ratio analysis of L lattice SAW generating function.

n	$r_n^{(0)}$	$l_n^{(0)}$	$q_n^{(0)}$	$r_n^{(1)}$	$l_n^{(1)}$	$q_n^{(1)}$	$r_n^{(2)}$	$l_n^{(2)}$	$q_n^{(2)}$	$r_n^{(3)}$	$l_n^{(3)}$	$q_n^{(3)}$
2	1.6266	1.3919	—	1.6148	1.4514	—	1.6193	1.5567	—	1.6176	1.6384	—
3	1.6097	1.5760	1.6680	1.6057	1.5874	1.6553	1.6017	1.5666	1.5716	1.5978	1.5581	1.5180
4	1.5987	1.5655	1.5551	1.5964	1.5688	1.5502	1.5952	1.5756	1.5846	1.5935	1.5809	1.6036
5	1.5924	1.5673	1.5699	1.5913	1.5707	1.5736	1.5899	1.5688	1.5587	1.5888	1.5697	1.5529
6	1.5883	1.5677	1.5685	1.5874	1.5680	1.5626	1.5865	1.5693	1.5701	1.5856	1.5696	1.5694
7	1.5852	1.5665	1.5634	1.5854	1.5673	1.5656	1.5838	1.5680	1.5648	1.5832	1.5687	1.5663
8	1.5828	1.5664	1.5661	1.5823	1.5668	1.5654	1.5818	1.5674	1.5657	1.5812	1.5679	1.5657
9	1.5810	1.5662	1.5656	1.5806	1.5666	1.5657	1.5801	1.5670	1.5657	1.5797	1.5674	1.5658
10	1.5795	1.5661	1.5656	1.5792	1.5664	1.5657	1.5788	1.5668	1.5658	1.5785	1.5671	1.5659
11	1.5783	1.5660	1.5657									

This may be compared with the earlier estimates of both Barber (1970) and Malakis (1975) of 1.733 ± 0.003.

For the L lattice, our results are shown in table 3. These data are particularly well behaved, and both the ratios $r_n^{(\alpha)}$ and the linear extrapolants $l_n^{(\alpha)}$ have 'settled down', and allow us to estimate

$$\mu_{\mathrm{L}} = 1.5658 \pm 0.0010.$$

Malakis's earlier estimate, based on a shorter series, was $\mu_{\mathrm{L}} = 1.559 \pm 0.003$, which we therefore consider unacceptably low.

We have also studied the above series by conventional Padé analysis methods. Firstly, Padé approximants to the logarithmic derivative of the SAW generating function were formed, and in order to save space we merely quote the results of an extensive series of tables. These give unbiased, simultaneous estimates of both μ and the critical exponent γ. For the honeycomb lattice, the approximants of the 34-term series appear to 'settle down' beyond the 27th term to yield $\mu = 1.8478 \pm 0.0002$ and $\gamma = 1.341 \pm 0.003$, in agreement with Nienhuis's exact results of $\mu = 1.847\,759\ldots$ and $\gamma = 1.343\,75$. For the Manhattan lattice we only have 28 terms, yet the approximants appear to 'settle down' beyond the 22nd term and suggest values of $\mu = 1.7339 \pm 0.0002$ and $\gamma = 1.321 \pm 0.003$. This critical point is in agreement with that found by ratio analysis, but the exponent is not in agreement with the value $\gamma = 1.343\,75$ claimed by Nienhuis to be exact for the honeycomb lattice. This suggests that either universality is violated, and that this non-regular lattice has a different exponent to the regular honeycomb lattice, or that the series is too short to display asymptotic behaviour. At this stage we incline strongly to the latter view. This view is reinforced by Padé analyses of other regular lattices, notably the square and triangular, which suggest values of γ of around 1.33, that is, midway between those found for the honeycomb and Manhattan lattice. It would be astonishing, if, say, the square and honeycomb lattice SAW generating function displayed different exponents, and accordingly we are led to the conclusion that the series are too short to display asymptotic behaviour. Examination of regular two-dimensional SAW series supports this view (Guttmann 1983) in that the diagonal and off-diagonal Padé approximants appear to converge in blocks. For example, the honeycomb lattice approximants at first appear to converge to $(1/\mu, \gamma) = (0.5410, 1.333)$, while higher-order approximants then settle down to the more accurate values of $(0.541\,18, 1.342)$.

Further, studies of Ising model and related series lead to the observation that any variation of homogeneous conditions usually slows down the rate of convergence of any extrapolation scheme. This also appears to have been the case here, where the lattices under consideration are non-regular.

Unfortunately, numerical difficulties have prevented us from obtaining Dlog Padé approximants for the L lattice. However, we have used the estimates of μ obtained by the ratio method in order to estimate γ by the usual technique of forming Padé approximants to $(x_c - x)(\mathrm{d}/\mathrm{d}x)\ln(f(x))|_{x=x_c} = 1/\mu$, where f is the SAW generating function. The results of this exercise suggest that $\gamma = 1.320$ (Manhattan), $\gamma = 1.343$ (honeycomb) and $\gamma = 1.350$ (L lattice).

Finally, if we assume that $\gamma = 1\frac{11}{32}$ for all two-dimensional lattices and form Padé approximants to $[f(x)]^{1/\gamma}$, which approximants should have poles at $x = 1/\mu$, we find $\mu = 1.733$ (Manhattan), $\mu = 1.8479$ (honeycomb) and $\mu = 1.5659$ (L lattice). (For both the Manhattan and L lattice overflow problems prevented all series coefficients from being used.)

The Padé results are therefore quite consistent with the ratio method estimates, the only disturbing feature being the apparent variation of exponent estimates with lattice, which we believe is an artifact of short series. Accordingly, it would be most valuable to extend the available series by several terms—both for the lattices considered here, and for the triangular and square lattices.

5. Conclusion

We have generalised Kesten's lemma to apply to non-regular two-dimensional lattices, and used both that extension and an application of Ahlberg and Janson's inequality to obtain the best extant upper and lower bounds on the connective constant of the Manhattan and L lattices.

Analyses of series expansions have given numerical estimates of these quantities, and our results may be summarised as

$$1.5353 < \mu \approx 1.565\,75 \pm 0.0005 < 1.5986 \qquad \text{(L lattice)}$$

and

$$1.6336 < \mu \approx 1.7340 \pm 0.0009 < 1.7912 \qquad \text{(Manhattan lattice).}$$

Acknowledgments

I would like to thank Professor J Keller for suggesting this problem, Professor S Whittington for numerous helpful discussions, Professor B Nienhuis for helpful communication and Professors M Van Dyke of Stanford University and J A Campbell of Exeter University for the hospitality of their departments where this work was carried out. Finally, I am indebted to Professors Ahlberg and Janson for providing their manuscript prior to publication.

References

Ahlberg R and Janson S 1982 *Preprint* Uppsala University
Barber M N 1970 *Physica* **48** 237
Fisher M E and Sykes M F 1959 *Phys. Rev.* **114** 45
Grassberger P 1982 *Z. Phys.* B *Condensed Matter* **48** 255
Guttmann A J 1983 Submitted to *J. Phys. A: Math. Gen.*
Guttmann A J and Whittington S G 1978 *J. Phys. A: Math. Gen.* **11** 721
Hammersley J M and Welsh D J A 1962 *Quart. J. Math. Oxford* (2) **13** 108
Kasteleyn P W 1963 *Physica* **29** 1329
Kesten H 1963 *J. Math. Phys.* **4** 960
Malakis A 1975 *J. Phys. A: Math. Gen.* **8** 1885
Nienhuis B 1982 *Phys. Rev. Lett.* **49** 1062
Wakefield A J 1951 *Proc. Camb. Phil. Soc.* **47** 799
Wall F T, Hiller J A and Atchison W F 1955 *J. Chem. Phys.* **23** 913
Wilker J B and Whittington S G 1979 *J. Phys. A: Math. Gen.* **12** L245